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PREDICTION OF THE EFFECTIVE ELASTIC PROPERTIES OF SPHEROPLASTICS BY THE GENERALIZED SELF-CONSISTENT METHOD

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The problem of predicting the effective elastic properties of composites with prescribed random location and radius variation in spherical inclusions is solved using the generalized self-consistent method. The problem is reduced to the solution of the averaged boundary-value problem of the theory of elasticity for a single inclusion with an inhomogeneous transition layer in a medium with desired effective elastic properties. A numerical analysis of the effective properties of a composite with rigid spherical inclusions and a composite with spherical pores is carried out. The results are compared with the known solution for the periodic structure and with the solutions obtained by the standard self-consistent methods.

Introduction. The effective physical and mechanical properties of composites are due to a complex interaction between a large number of the elements which form the structure of the material. The irregular character of real structures requires solving the problems of predicting the effective physical and mechanical properties of composites in a statistic formulation. Pan'kov [1, 2] illustrated the potentials of the generalized self-consistent method by referring to the prediction of the effective elastic properties of unidirectional fibrous composites. We consider the application of this method to composites with random structures of spherical inclusions.

Generalized Self-Consistent Method. Let a random structure of a composite possess homogeneity and ergodicity properties [3–5] and be given by the characteristic realization in a certain domain V with boundary Γ. All inclusions have the same elastic properties, geometrical form, and space orientation. The ideal contact conditions hold at the interphase surfaces. There is a statistic spread only in the relative location and dimensions of inclusions and in the similarity coefficients α. The tensor of elastic properties $C(\boldsymbol{r})$ in the domain V admits the representation $C_{ijmn}(\boldsymbol{r}) = \omega(\boldsymbol{r})C^F_{ijmn} + (1-\omega(\boldsymbol{r}))C^M_{ijmn}$ in terms of the inclusion indicator function

$$\omega(\boldsymbol{r}) = \begin{cases} 1, & \boldsymbol{r} \in V_F, \\ 0, & \boldsymbol{r} \notin V_F, \end{cases} \tag{1}$$

where V_F is the inclusion domain in the domain V; the elastic properties of the inclusions and the matrix are homogeneous and are specified by the elastic-property tensors C^F and C^M, respectively.

The generalized self-consistent method [1, 2] allows one to reduce the problem of predicting the tensor of the effective elastic properties C^* of a composite, which consists of solving the boundary-value problem of the theory of elasticity for a microinhomogeneous domain V [3–6]

$$\frac{\partial}{\partial r_j}\Big(C_{ijmn}(\boldsymbol{r})\,\frac{\partial}{\partial r_n}\,u_m(\boldsymbol{r})\Big) = 0 \tag{2}$$

[e.g., relative to the displacement field $\boldsymbol{u}(\boldsymbol{r})$ with boundary conditions of the form $u_i|_\Gamma = \varepsilon^*_{ij} r_j$], to the simpler averaged problem

$$\frac{\partial}{\partial \xi_j}\Big(a_{ijmn}(\xi)\,\frac{\partial}{\partial \xi_n}\,\bar{u}_m(\xi)\Big) = 0 \tag{3}$$

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for the displacements $\bar{u}_i = \varepsilon^*_{ij}\xi_j$ prescribed sufficiently far from the origin of coordinates ξ. Here ε^* is the specified tensor of homogeneous small elastic macrostrain of the composite and the coordinate axes ξ_i are oriented along the r_i axes. Formally, the relation between the actual $\boldsymbol{u}(\boldsymbol{r})$ and the averaged $\bar{\boldsymbol{u}}(\xi)$ fields of the boundary-value problems (2) and (3) is found by averaging

$$\bar{u}_i(\xi) = \frac{1}{N}\sum_{k=1}^{N} \alpha_{(k)}^{\beta-1}(u_i(\boldsymbol{r}_{(k)} + \alpha_{(k)}\xi) - u_i(\boldsymbol{r}_{(k)})).$$

Here N is the number of inclusions in the characteristic domain $\Omega \subset V$ at a sufficiently large distance from the boundary Γ, $\boldsymbol{r}_{(k)}$ are the radius-vectors of the center of the kth inclusion in the domain Ω, $\alpha_{(k)}$ is the similarity coefficient of the kth inclusion which is specified relative to the dimensions of the formal inclusion v of the averaged problem and the volume of which is $|v| = \frac{1}{N}\sum_{k=1}^{N}|v_{(k)}|$, where $|v_{(k)}|$ is the volume of the kth inclusion, and the power β depends on the dimensionality of the problem and is equal to 1, 2, or 3 for laminated, unidirectional fibrous, or granular composites, respectively.

The field of elastic properties of the averaged problem (3)

$$a_{ijmn}(\xi) = \left[\frac{1-\bar{\omega}(\xi)}{1-v_F}\,C^M_{ijgh} + \left(\frac{\bar{\omega}(\xi)}{v_F}\,C^F_{ijqp} - \frac{1-\bar{\omega}(\xi)}{1-v_F}\,C^M_{ijqp}\right)S^{-1}_{qpdb}(C^*_{dbgh} - C^M_{dbgh})\right]k^{-1}_{ghmn}(\xi) \tag{4}$$

takes into account the near order of relative arrangement and the variations in the dimensions of the inclusions by means of a special averaged indicator function

$$\bar{\omega}(\xi) = \frac{1}{N}\sum_{k=1}^{N}\alpha^{\beta}_{(k)}\omega(\boldsymbol{r}_{(k)} + \alpha_{(k)}\xi),$$

which is calculated using the specified field of the indicator function $\omega(\boldsymbol{r})$ [see (1)] of the composite. In formula (4), $k^{-1}(\xi)$ is the tensor field inverse to the field

$$k_{ijmn}(\xi) = \frac{1-\bar{\omega}(\xi)}{1-v_F}\,E_{ijmn} + \frac{\bar{\omega}(\xi)-v_F}{v_F(1-v_F)}\,S^{-1}_{ijdb}(C^*_{dbmn} - C^M_{dbmn}),$$

where E is the identity tensor, S^{-1} is the tensor inverse to the difference tensor $S \equiv C^F - C^M$, and v_F is the volume proportion of the inclusions in the composite.

If C^F and C^M are isotropic and C^* is the isotropic or transversely isotropic tensor, the field $a(\xi)$ is an isotropic or transversely isotropic field, respectively, at each point ξ. The bulk modulus $K_a(\xi)$ and the shear modulus $G_a(\xi)$ for the isotropic field $a(\xi)$ or, respectively, the bulk modulus for the plane strain $k_{a12}(\xi)$ and the shear modulus $G_{a12}(\xi)$ in the plane of isotropy (e.g., $\xi_1 O \xi_2$) for the case of the transversely isotropic field $a(\xi)$, can be determined by formula (4) using a unified relation of the form

$$L_a(\xi) = L_M\,\frac{\beta(\xi) + \left(\frac{L_F}{L_M}\,\alpha(\xi) - \beta(\xi)\right)\gamma_L}{\beta(\xi) + (\alpha(\xi) - \beta(\xi))\gamma_L}, \tag{5}$$

where

$$\alpha(\xi) = \frac{\bar{\omega}(\xi)}{v_F}, \qquad \beta(\xi) = \frac{1-\bar{\omega}(\xi)}{1-v_F}, \qquad \gamma_L = \frac{L^* - L_M}{L_F - L_M}. \tag{6}$$

The symbol L in formulas (5) and (6) should be replaced in succession by K and G or k_{12} and G_{12}, respectively. For an arbitrary, transversely isotropic tensor of elastic properties C with the plane of isotropy $\xi_1 O \xi_2$, the relations $k_{12} = (1/2)(C_{1111} + C_{1122})$ and $G_{12} = C_{1212}$ hold.

Thus, from the solution of the auxiliary averaged problem (3), (4) for the displacement field $\bar{u}_i(\xi) \equiv \bar{U}_{imn}(\xi)\varepsilon^*_{mn}$, one can determine the tensor $\bar{U}^F(C^*)$ as a function of C^*, whose components have the form

$$\bar{U}^F_{ijmn} = \frac{1}{|v|}\int_v \frac{\partial}{\partial \xi_j}\,\bar{U}_{imn}(\xi)\,dv.$$

The solution of the equation $C^*_{ijmn} = C^M_{ijmn} + v_F(C^F_{ijdb} - C^M_{ijdb})\bar{U}^F_{dbmn}(C^*)$ is the desired tensor of effective elastic properties C^* of the composite with a random structure.

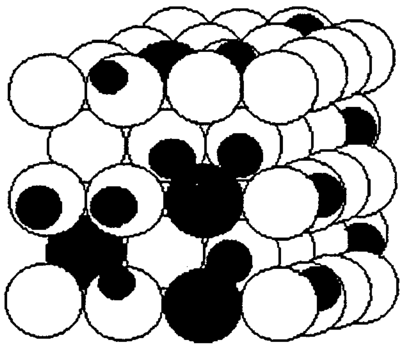

Fig. 1

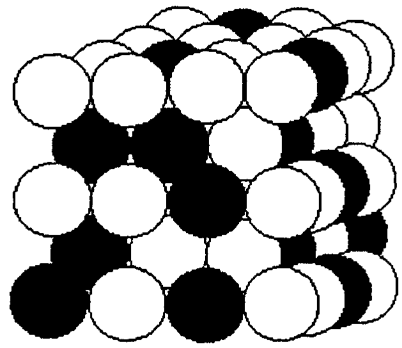

Fig. 2

Numerical Analysis. We consider the numerical results and compare them with known calculations [4–7] of the effective isotropic elastic properties of composites on the basis of two types of the space quasiperiodic model of real random structures with spherical inclusions. The models of random structures (Figs. 1 and 2) are based on the hexagonal close packing of spherical cells; the probability of appearance of inclusion-free cells is calculated based on the volume proportion of the inclusions v_F in the composite and on the guaranteed minimum thickness, which is assumed to be 2% of the radius of the cell. For the model of the first type (Fig. 1), we also specify the variation coefficient equal, for example, to 0.1 for α (α is the similarity factor of the dimensions of inclusions) and random displacements of the inclusions inside the cells. For the model of the second type (Fig. 2), all inclusions are of the same dimensions and have no displacements inside the cells. The maximal dimensions of the inclusions in both models and the dimensions of the spherical cells are the same.

Within the framework of the calculation scheme of the averaged problem, we solve two problems for a spherical inclusion v with the coordinate origin ξ at its center. The inclusion is surrounded by a spherical transition layer, which is inhomogeneous with respect to the variable $\xi \equiv |\xi|$ but isotropic at each point and has a field of elastic properties $a(\xi)$, and by a homogeneous isotropic medium with the tensor of elastic properties C^*. The problems are solved under the following simple loading conditions of the medium at infinity: 1) the three-dimensional tension; 2) the shear, for example, in the $\xi_1 O \xi_2$ plane. Solving these problems simultaneously, one can determine the effective bulk modulus K^* and the shear modulus G^* of the composite.

In a numerical solution, the transition layer of thickness $7r_F$ (r_F is the radius of the central inclusion v) is divided into 50 thin layers, and the elastic properties of each layer are assumed to be homogeneous and isotropic. The general solutions for the deformation fields of each thin layer are known [7]. The ideal contact conditions allow one to formulate a system of linear algebraic equations of the band type for desired coefficients.

The values of the normalized effective bulk modulus $k = K^*/K_M$ and the shear modulus $g = G^*/G_M$ (K_M and G_M are the bulk modulus and the shear modulus of the matrix) of the two models of random structures predicted numerically by the generalized self-consistent method are given with the superscripts (1) and (2) in the first two rows of Table 1. Calculations were performed for the following values of the Young's modulus and Poisson ratio of the inclusions: $E_F = 100E_M$ and $\nu_F = 0.2$ and $E_F = 0$ and $\nu_F = 0$. In both cases, the Young's modulus $E_M = 3.75$ GPa and the Poisson ratio is $\nu_M = 0.4$ for an isotropic epoxy matrix, the subscript R refers to the first-approximation solutions of Vanin [4] for a periodic structure, the subscripts 1 and 2 denote the solutions obtained by the generalized self-consistent method with the use of piecewise-constant approximations

$$1)\ \ \bar{\omega}(\xi) = \begin{cases} 1, & 0 \leqslant \xi \leqslant r_F, \\ 0, & r_F < \xi \leqslant r_0, \\ v_F, & \xi > r_0; \end{cases} \qquad 2)\ \ \bar{\omega}(\xi) = \begin{cases} 1, & 0 \leqslant \xi \leqslant r_F, \\ v_F, & \xi > r_F \end{cases} \tag{7}$$

TABLE 1

Type of the model	$v_F = 0.2$		$v_F = 0.3$		$v_F = 0.4$		$v_F = 0.5$		$v_F = 0.6$	
	I	II	I	II	I	II	I	II	I	II
$k^{(1)}$	1.313	0.456	1.533	0.328	1.796	0.250	2.113	0.197	—	—
$k^{(2)}$	1.329	0.423	1.586	0.280	1.938	0.188	2.397	0.131	2.948	0.097
$k_R = k_1 = k_-$	1.306	—	.1.521	—	1.804	—	2.191	—	2.754	—
$k_R = k_1 = k_+$	—	0.471	—	0.342	—	0.250	—	0.182	—	0.129
k_2	1.355	0.378	1.702	0.203	2.436	0.082	4.507	—	8.991	—
k_+	4.688	—	6.846	—	9.264	—	11.991	—	15.092	—
$g^{(1)}$	1.614	0.671	2.092	0.536	2.681	0.428	3.401	0.343	—	—
$g^{(2)}$	1.676	0.652	2.303	0.495	3.266	0.361	4.610	0.258	6.312	0.184
g_R	1.483	0.690	1.827	0.565	2.282	0.455	2.914	0.357	3.848	0.270
g_1	1.591	0.678	2.078	0.539	2.794	0.414	3.863	0.305	5.504	0.215
g_2	1.751	0.627	2.680	0.429	4.980	0.221	12.177	—	28.203	—
g_-	1.549	—	1.938	—	2.453	—	3.166	—	4.218	—
g_+	13.950	0.690	21.556	0.565	30.102	0.455	39.775	0.357	50.813	0.270

Note. I) $E_F = 100E_M$ and $\nu_F = 0.2$; II) $E_F = 0$ and $\nu_F = 0$.

for the averaged indicator function $\bar{\omega}(\xi)$, where the parameter r_0 is determined from the relation $(r_F/r_0)^3 = v_F$, and the subscripts minus and plus denote Khashin–Shtrikman's boundary values for a macroscopic two-phase medium [5–7]. For a composite with spherical pores, the equality $k_- = g_- = 0$ holds. The solutions for k_1 and g_1 and k_2 and g_2 obtained by means of the piecewise-constant approximations (7) are identical to the solutions obtained by the known self-consistent methods [5–7].

Conclusions. The generalized self-consistent method allowed us to reduce the problem of predicting the effective elastic properties of composite materials of random space structures to the solution of a simpler averaged problem for an inclusion with a transition layer in a medium with the desired effective elastic properties. The characteristic dimension of the transition layer is determined by the correlation radius of the random structure, and its elastic properties take into account the near order of relative arrangement of the inclusions.

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